Physics 566

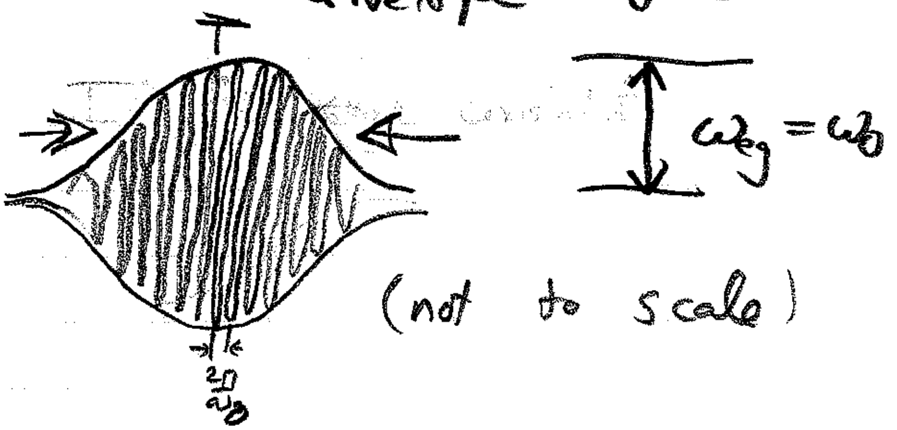
Problem Set #4 Solutions

Problem 1: Free induction decay

(a) Atom initial in its ground state.
Pulse of light of duration $T \gg \frac{1}{\omega_0}$

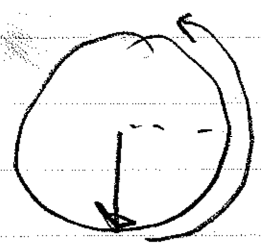
$\Rightarrow$ slowly varying envelope $E(t)e^{-i\omega_0 t}$ $\nwarrow$ carrier

$\swarrow$ on-resonance



(not to scale)

On-resonance (ignoring finite bandwidth due to finite duration)

Bloch sphere  $\theta(t)$ (rotation angle on Resonance)

$$\dot{\theta}(t) = \Omega(t) = \frac{E(t)\, d_{eg}}{\hbar}$$ (Instantaneous Rabi frequency)

Rotation angle $\int_0^T \Omega(t)\,dt = \Theta$ $\Rightarrow$ (Next Page)

To achieve $\rho_{ee} = \frac{1}{2} \Rightarrow \Theta = \frac{\pi}{2}$

$\Rightarrow$ Condition on 'pulse-area' for $\frac{\pi}{2}$-pulse

$$\boxed{\int_0^T E(t)\,dt = \frac{\pi\hbar}{2 d_{eg}}}$$

Assuming phase of pulse $= 0$ in rotating frame

After pulse, $|\tilde{\psi}\rangle = \cos\frac{\theta}{2}|g\rangle - i\sin\frac{\theta}{2}|e\rangle$

$\Theta = \frac{\pi}{2} \Rightarrow |\tilde{\psi}\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle - i|e\rangle\right)$

$|\tilde{\psi}\rangle\langle\tilde{\psi}| = \tilde{\rho} \Rightarrow \boxed{\tilde{\rho}_{eg} = \langle g|\psi\rangle\langle\psi|e\rangle = \frac{i}{2}}$

(b) For ^{23}Na, first excited state, $\Gamma^{-1} = 16\,ns$, wavelength of transition $\lambda_0 = 589$ nm.

Take pulse duration $T = 100\,ps \ll \Gamma^{-1}$

Fourier transform-limited bandwidth

$$\Delta\omega \sim \frac{2\pi}{100\,ps} = 6\times10^{-10}\,s^{-1}$$

$$\omega_0 = \frac{2\pi c}{\lambda_0} = 3.2\times10^{15}\,s^{-1} \gg \Delta\omega$$

$\Rightarrow$ Pulse envelope slowly varying

Constant Amplitude Square $\frac{\pi}{2}$ - pulse

$$\Rightarrow \quad \Theta = \underset{\nwarrow \text{ Rabi - frequency}}{\Omega T} = \frac{\pi}{2}$$

To find the intensity use saturation parameter

on resonance: $s = I/I_{sat} = \frac{2\Omega^2}{\Gamma^2} = \frac{2\,\Theta^2}{(\Gamma T)^2}$

$$\Rightarrow \quad I = \frac{\pi^2}{2(\Gamma T)} I_{sat}$$

$$I_{sat} = \frac{\hbar\omega_0}{\sigma_0}\frac{\Gamma}{2} \qquad \sigma_0 = \frac{3}{2\pi}\lambda^2 = 1.65 \times 10^{-9}\,cm^2$$

resonant cross-section

$$\hbar\omega_0 = \frac{hc}{\lambda} = 3.37 \times 10^{-19}\,J = 2eV$$

$$\Rightarrow \quad \underbrace{I_{sat} \cong 6 \frac{mW}{cm^2}}$$

$$\therefore \quad I = \frac{(3.14)^2}{2\left(\frac{100\times10^{-12}s}{16\times10^{-9}s}\right)^2} 6\frac{mW}{cm^2}$$

$$\Rightarrow \boxed{I \approx 800 \;W/cm^2}$$

Intensity @ which $\Omega = \Gamma \quad \Rightarrow \quad I = 2 I_{sat}$

$$\Rightarrow \quad \boxed{I \cong 12 \;mW/cm^2}$$

(c) After the atom is placed in a superposition of $|g\rangle$ and $|e\rangle$, the dipole will oscillate @ frequency $\omega_{eg} = \omega_0$. In doing so, it will spontaneously decay back to ground state. This is known in "NMR" as "free-induction decay".

To find the evolution, solve the Optical Bloch equations for the initial density operator

$$\rho_{ee} = \rho_{gg} = \frac{1}{2} \qquad \rho_{eg} = \frac{i}{2}$$

$$\Rightarrow \quad w = \rho_{ee} - \rho_{gg} = 0 \qquad u = 0 \qquad v = \frac{1}{2}$$

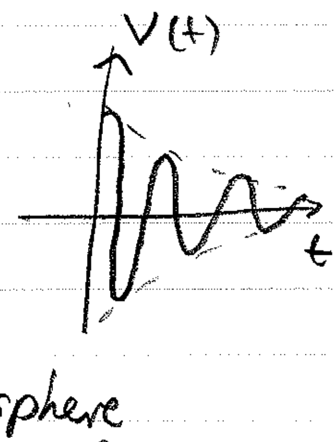
In rotating frame $\dot{\tilde{u}} = -\frac{\Gamma}{2}\tilde{u}$, $\dot{\tilde{v}} = -\frac{\Gamma}{2}\tilde{v}$, $\dot{\tilde{w}} = -\Gamma w - \frac{\Gamma}{2}$

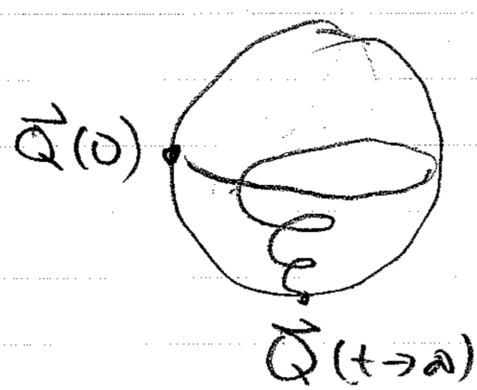
$$\Rightarrow \quad \tilde{u}(t) = \tilde{u}(0)e^{-\frac{\Gamma}{2}t} = 0$$

$$\tilde{v}(t) = \tilde{v}(0)e^{-\frac{\Gamma}{2}t} = \frac{1}{2}e^{-\frac{\Gamma}{2}t}$$

$$\tilde{w}(t) = \frac{1}{2}\left(e^{-\Gamma t} - 1\right)$$

In lab frame $\quad v(t) = \tilde{v}(t)\sin\omega t$

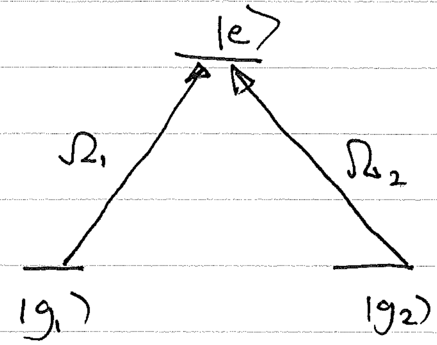




On Bloch sphere trajector spirals in from equator to ground state @ pole.

Problem 2: Dark States

(a) The Hamiltonian for the "lambda" 3-level atom



$$\hat{H}_{AL} = -\frac{\hbar}{2}\left[\Omega_1\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \Omega_2\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)\right]$$

$$= -\frac{\hbar}{2}\begin{array}{cc}\begin{bmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{bmatrix} & \begin{array}{c} |g_1\rangle \\ |g_2\rangle \\ |e\rangle \end{array}\end{array}$$ (matrix representation)

Eigenvalues: Find characterist polynomial

$$\det(\lambda \hat{\mathbb{1}} - \hat{H}) = 0$$

$$\Rightarrow \lambda^3 - \lambda\left(\frac{\hbar\Omega_1}{2}\right)^2 - \lambda\left(\frac{\hbar\Omega_2}{2}\right)^2 = 0$$

$\Rightarrow$ Three energy eigenvalues

$$\underline{\lambda_{\pm} = \pm\frac{\hbar}{2}\sqrt{\Omega_1^2 + \Omega_2^2}}$$

$$\underline{\lambda_D = 0}$$

Eigenvectors:

$\lambda_D = 0 \Rightarrow \begin{bmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{bmatrix} \begin{bmatrix} a_D \\ b_D \\ c_D \end{bmatrix} = 0$

$$\Rightarrow \boxed{|\lambda\rangle = \Omega_2 |g_1\rangle - \Omega_1 |g_2\rangle \quad \text{(unnormalized)}}$$

$$\lambda_\pm \Rightarrow \begin{bmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{bmatrix} \begin{bmatrix} a_\pm \\ b_\pm \\ c_\pm \end{bmatrix} = \pm \frac{\hbar}{2} \sqrt{\Omega_1^2 + \Omega_2^2} \begin{bmatrix} a_\pm \\ b_\pm \\ c_\pm \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_\pm \\ b_\pm \\ c_\pm \end{bmatrix} = \begin{bmatrix} \pm\Omega_1 \\ \pm\Omega_2 \\ \sqrt{\Omega_1^2+\Omega_2^2} \end{bmatrix}$$

$$\boxed{|\lambda_\pm\rangle = \pm\Omega_1 |g_1\rangle \pm \Omega_2 |g_2\rangle + \sqrt{\Omega_1^2+\Omega_2^2}\,|e\rangle \quad \text{(unnormalized)}}$$

These are the dressed states

$E = +\frac{\hbar}{2}\sqrt{\Omega_1^2+\Omega_2^2}$ —— $|\lambda_+\rangle$

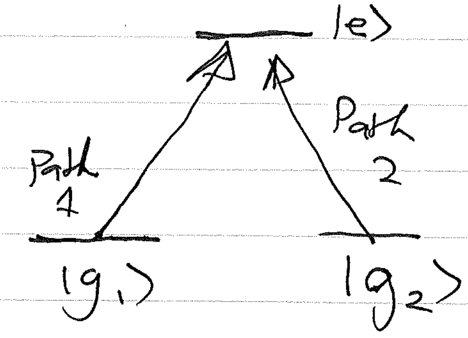
$E = 0$ —— $|\lambda_D\rangle$

$E = -\frac{\hbar}{2}\sqrt{\Omega_1^2+\Omega_2^2}$ —— $|\lambda_-\rangle$

The state $|\lambda_D\rangle = \Omega_2 |g_1\rangle - \Omega_1 |g_2\rangle$ is dark in that $\hat{H}_{AL} |\lambda_D\rangle = 0$

$\Rightarrow$ **If the atom is in this superposition of ground states it doesn't absorb light**

How is this possible? **Answer: quantum interference**



There are "two paths" to get from the ground manifold to the excited state. If these paths destructively interfere, then the atom cannot absorb the light. Thus the anti-symmetric combination of $|g_1\rangle$ and $|g_2\rangle$ weighted by approximated amplitude is dark

$$\hat{H}_{AL}\left(c_1|g_1\rangle + c_2|g_2\rangle\right) = \left(c_1\Omega_1 + c_2\Omega_2\right)|e\rangle$$

If $c_1\Omega_1 + c_2\Omega_2 = 0 \Rightarrow$ destructive interference

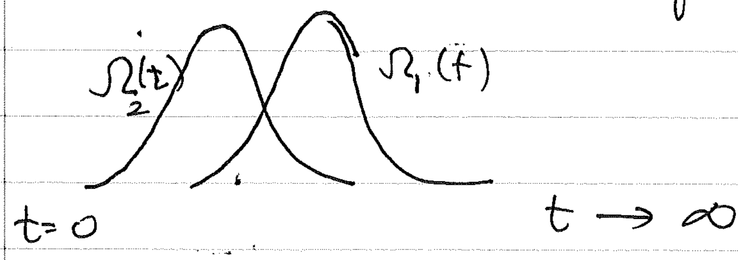
$$\Rightarrow \frac{c_1}{c_2} = \frac{-\Omega_2}{\Omega_1}$$

(b) Adiabatic transfer via "nonintuitive pulse sequence".

Goal: Start with atom in $|g_1\rangle$ and transfer to $|g_2\rangle$

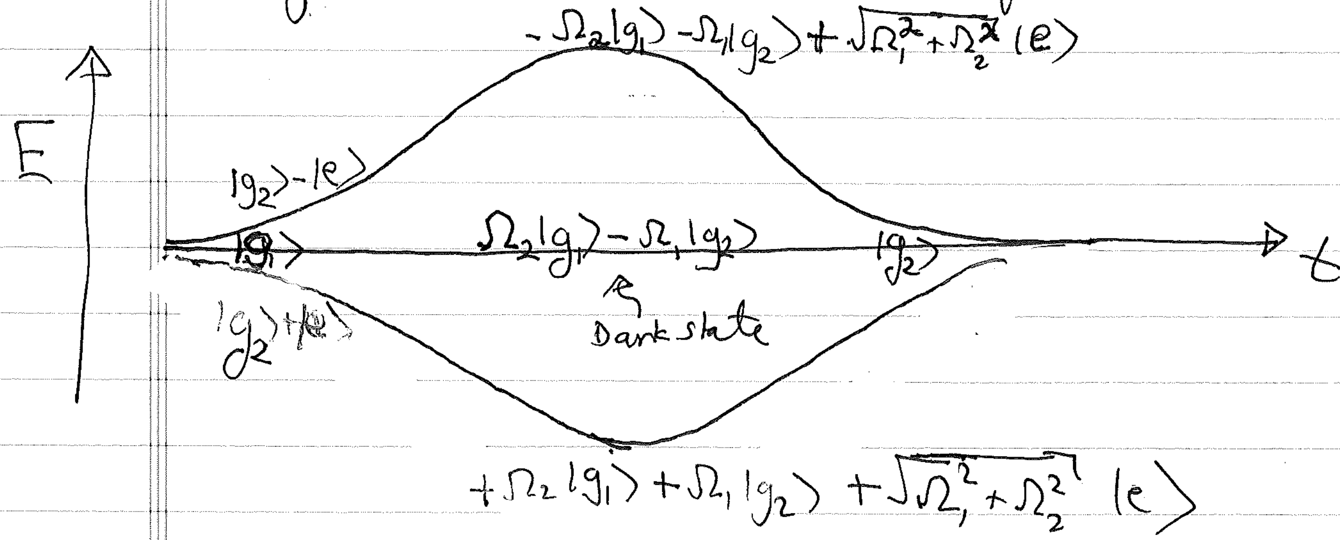
Protocol: Adiabatically "rotate" $|g_1\rangle$ to $|g_2\rangle$ by slowly changing the dark state

Non-intuitive pulse sequence



First apply a pulse ~~changing~~ driving $|g_2\rangle \to |e\rangle$. For this field $|g_1\rangle$ is dark. The "morph" toward a pulse driving $|g_1\rangle \to |e\rangle$. In this configuration $|g_2\rangle$ is dark. If this is all done adiabatically, we transfer $|g_1\rangle \to |g_2\rangle$

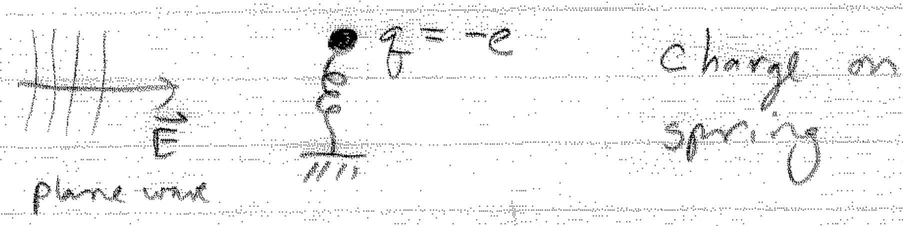
We see this via the dressed energy-level diagram as a function of time:



Adiabatic population transfer from

$$|g_1\rangle \to |g_2\rangle$$

Problem 3: Lorentz Classical Model



Eq of motion $m\ddot{\vec{x}} = -m\Gamma\dot{\vec{x}} - m\omega_0^2\vec{x} - e\vec{E}(t)$

Rate at which field does work

$$\frac{dW}{dt} = \dot{\vec{x}} \cdot \vec{F} = -e\dot{\vec{x}} \cdot \vec{E}$$

Need to solve for $\dot{\vec{x}}$ in steady state.

Use complex representation: $\vec{E} = \vec{E}_0 e^{-i\omega t}$

$\Rightarrow$ steady state $\vec{x} = \vec{x}_0 e^{-i\omega t}$

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^2\vec{x} = -\frac{e}{m}\vec{E}_0 e^{-i\omega t}$$

$$\Rightarrow \left(-\omega^2 + \omega_0^2 - i\omega\Gamma\right)\vec{x}_0 = -\frac{e}{m}\vec{E}_0$$

$$\Rightarrow \vec{x}_0 = \left(\frac{-e/m}{-\omega^2 + \omega_0^2 - i\omega\Gamma}\right)\vec{E}_0$$

Go to near resonance limit $\omega - \omega_0 \equiv \Delta$

$$\Rightarrow \omega_0 = \omega - \Delta \qquad \omega_0^2 - \omega^2 = (\omega-\Delta)^2 - \omega^2$$

$$\simeq -2\omega\Delta \text{ to } \mathcal{O}(\Delta)$$

$$\Rightarrow \vec{x}_0 \approx \left(\frac{e/2m\omega}{\Delta + i\frac{\Gamma}{2}} \right) \vec{E}_0$$

$\therefore$ Time averaged over a period

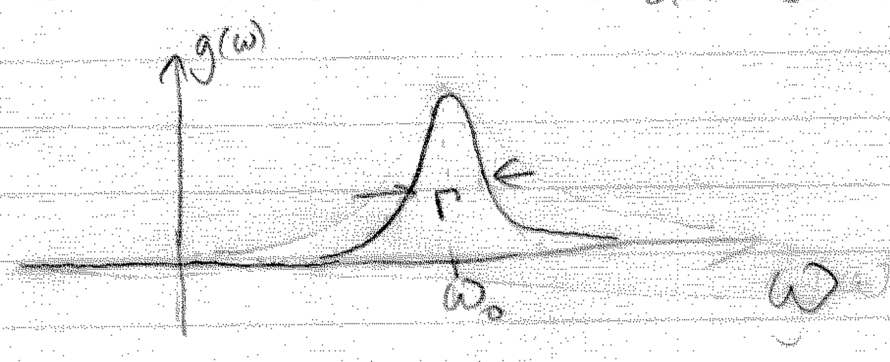
$$\frac{dW}{dt} = \frac{-e}{2} \mathrm{Re}\left(-i\omega \vec{x}_0 \cdot \vec{E}_0\right)$$

$$= \frac{-e\omega}{2} \mathrm{Im}\left(\vec{x}_0 \cdot \vec{E}_0\right)$$

$$= -\frac{e\omega}{2}\left(\frac{e}{2m\omega}\right) \frac{\frac{\Gamma}{2}}{\Delta^2 + \frac{\Gamma^2}{4}} |\vec{E}_0|$$

$$= \frac{\pi e^2}{4m} \underbrace{\left(\frac{\frac{\Gamma}{2\pi}}{\Delta^2 + \frac{\Gamma^2}{4}} \right)}_{g(\omega)} |\vec{E}_0|^2$$

$g(\omega)$ ← atomic line-shape



(b) Absorption cross-section

$$P_{abs} = \sigma_{abs} I_{inc} = \sigma_{abs} \frac{c}{8\pi} |\vec{E}_0|^2$$

$$\| $$

$$\frac{dW_{abs}}{dt} = \frac{\pi e^2}{4m} g(\omega) |\vec{E}_0|$$

$$\Rightarrow \boxed{\sigma_{abs} = \frac{2\pi^2 e^2}{mc} g(\omega)}$$

Evaluate for $\frac{\Gamma}{2\pi} = 10\ MHz$ $\lambda = 589\ nm$

on resonance $\Delta = 0$ $g(\omega = \omega_0) = \frac{2}{\pi}\frac{1}{\Gamma}$

$$\Rightarrow \sigma_{abs} = 2\left(\frac{2\pi}{\Gamma}\right)\frac{e^2}{mc} = 2\left(\frac{2\pi}{\Gamma}\right) c \underbrace{\left(\frac{e^2}{mc^2}\right)}_{\text{classical electron radius} = r_e}$$

$$= 2\left(10^{-7} s^{+1}\right)\left(3\times 10^{10} \frac{cm}{s}\right)\left(2.8\times 10^{-13}\ cm\right)$$

$$\boxed{\sigma_{abs} = 1.68\times 10^{-9}\ cm^2}$$

Note: On resonance $\sigma_{abs} \sim \lambda_0^2 = 3.5\times 10^{-9}\ cm^2$

(c) $\sigma_{quantum} = 4\pi^2 \frac{e^2}{\hbar c} |\langle e|\vec{x}|g\rangle|^2 \omega\, g(\omega)$

Oscillator strength $f = \left.\frac{\sigma_{quantum}}{\sigma_{classical}}\right|_{resonance}$

$$\Rightarrow \boxed{f = \frac{2m\omega_0}{\hbar} |\langle e|\vec{x}|g\rangle|^2}$$

(d) The driven oscillator will radiate electromagnetically

$$P_{radiated} = \frac{ck^4}{3} |\vec{d}_0|^2 \quad \text{Larmor formula}$$

$$= \frac{ck^4 e^2}{3} |\vec{x}_0|^2$$

$$P_{abs} = -\frac{e\omega}{2} \operatorname{Im}(\vec{x}_0 \cdot \vec{E}_0)$$

Now $\vec{x}_0 = \alpha \vec{E}_0$ where $\alpha = \frac{e/2m\omega}{\Delta + i\frac{\Gamma}{2}}$

$$\Rightarrow \frac{ck^4 e^2}{3} |\vec{E}_0|^2 \underbrace{|\alpha|^2}_{\downarrow\ \frac{\frac{e^2}{4m^2\omega^2}}{\Delta^2 + \frac{\Gamma^2}{4}}} = -\frac{e\omega}{2} |\vec{E}_0|^2 \underbrace{\operatorname{Im}(\alpha)}_{\downarrow\ \frac{-i\frac{\Gamma}{2}\, e/2m\omega}{\Delta^2 + \frac{\Gamma^2}{4}}}$$

$$\therefore \boxed{\Gamma_{class} = \frac{2}{3}\frac{ck^4}{\omega^2}\frac{e^2}{m} = \frac{2}{3}\frac{e^2}{mc^3}\omega^2}$$

(e) We have quantum mechanically,

$$\Gamma_{quantum} = \frac{4}{3}\frac{k^3}{\hbar}|\langle e|\vec{d}_0|g\rangle|^2$$

$$= \frac{4}{3}\frac{e^2}{\hbar}|\langle e|\vec{x}_0|g\rangle|^2 \frac{\omega^3}{c^3}$$

$$\rightarrow \frac{\Gamma_{quant}}{\Gamma_{class}} = \frac{2m\omega}{\hbar}|\langle e|\vec{x}_0|g\rangle|^2 = f \checkmark$$

the oscillator strength

Problem 4: Radiation Reaction + decay

Two-level atom + Vacuum (RWA)

$$\hat{H} = \frac{1}{2}\hbar\omega_{eg}\,\hat{\sigma}_z + \sum_{\vec{k},\lambda}\hbar\omega_k\,\hat{a}^\dagger_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda}$$
$$- \sum_{\vec{k},\lambda}\hbar\left(g_{\vec{k}\lambda}\,\hat{\sigma}_+\hat{a}_{\vec{k}\lambda} + g^*_{\vec{k}\lambda}\,\hat{\sigma}_-\hat{a}^\dagger_{\vec{k}\lambda}\right)$$

where $g_{\vec{k},\lambda} = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\;\vec{\epsilon}_{\vec{k}\lambda}\cdot\vec{d}_{eg}$ (Vacuum Rabi freq)

(a) Heisenberg equations of motion

$$\frac{d}{dt}\hat{a}_{\vec{k},\lambda} = \frac{-i}{\hbar}\left[\hat{a}_{\vec{k},\lambda}, \hat{H}\right]$$

Notes: $\left[\hat{a}_{\vec{k},\lambda}, \hat{a}^\dagger_{\vec{k}',\lambda'}\right] = \delta_{\vec{k}\vec{k}'}\,\delta_{\lambda\lambda'}$

all other operators commute with $\hat{a}_{\vec{k},\lambda}$

$$\Rightarrow \boxed{\frac{d}{dt}\hat{a}_{\vec{k},\lambda} = -i\omega_k\,\hat{a}_{\vec{k},\lambda} + i\,g_{\vec{k},\lambda}\,\hat{\sigma}_-}$$

$$\frac{d}{dt}\hat{\sigma}_- = -\frac{i}{\hbar}\left[\hat{\sigma}_-, \hat{H}\right], \qquad \frac{d}{dt}\hat{\sigma}_z = -\frac{i}{\hbar}\left[\hat{\sigma}_z, \hat{H}\right]$$

Notes: $\left[\hat{\sigma}_+, \hat{\sigma}_-\right] = \hat{\sigma}_z$ $\qquad \left[\hat{\sigma}_z, \hat{\sigma}_\pm\right] = \pm 2\hat{\sigma}_\pm$

(Next Page)

$$\Rightarrow \boxed{\begin{aligned} \frac{d}{dt}\hat{\sigma}_- &= -i\omega_{eg}\hat{\sigma}_- - i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda}\,\hat{\sigma}_z \\ \frac{d}{dt}\hat{\sigma}_z &= 2i\sum_{\vec{k},\lambda}\left(g_{\vec{k}\lambda}\hat{\sigma}_+ a_{\vec{k}\lambda} - g_{\vec{k}\lambda}\hat{\sigma}_- a^\dagger_{\vec{k}\lambda}\right) \end{aligned}}$$

(b) Solve $\underbrace{\frac{d}{dt}\hat{a}_{\vec{k}\lambda} + i\omega_k a_{\vec{k}\lambda}}_{\text{Homogeneous eq}} = \underbrace{i g^*_{\vec{k}\lambda}\hat{\sigma}_-}_{\text{Source}}$

Use Green's function for homogeneous eq

$$\Rightarrow \hat{a}_{\vec{k}\lambda}(t) = \underbrace{\hat{a}_{\vec{k}\lambda}(0)e^{-i\omega_k t}}_{\text{vacuum}} + \underbrace{i g^*_{\vec{k},\lambda}\int_0^t \hat{\sigma}_-(t')\,e^{-i\omega_k(t-t')}dt'}_{\text{radiation from souce}}$$

(c) Equal-time commutator

$$[\hat{a}_{\vec{k},\lambda}(t), \hat{a}^\dagger_{\vec{k}',\lambda'}(t)] = [\hat{U}^\dagger(t)\hat{a}_{\vec{k}\lambda}(0)\hat{U}(t), \hat{U}^\dagger(t)\hat{a}^\dagger_{\vec{k}\lambda}(0)\hat{U}(t)]$$

$$= U^\dagger(t)\hat{a}_{\vec{k}\lambda}(0)\overbrace{\hat{U}(t)\hat{U}^\dagger(t)}^{=1}a^\dagger_{\vec{k}\lambda}(0)\hat{U}(t)$$

$$- U^\dagger(t)\hat{a}^\dagger_{\vec{k},\lambda}(0)\underbrace{\hat{U}(t)\hat{U}^\dagger(t)}_{=1}\hat{a}_{\vec{k}\lambda}(0)\hat{U}(t)$$

$$\Rightarrow [\hat{a}_{\vec{k}\lambda}(t), \hat{a}^\dagger_{\vec{k}'\lambda'}(t)] = \hat{U}^\dagger(t) \underbrace{[\hat{a}_{\vec{k}\lambda}(0), \hat{a}^\dagger_{\vec{k}'\lambda'}(0)]}_{\delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}} \hat{U}(t)$$

$$\Rightarrow \boxed{[\hat{a}_{\vec{k}\lambda}(t), \hat{a}^\dagger_{\vec{k}'\lambda'}(t)] = \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'} = [\hat{a}_{\vec{k}\lambda}(0), \hat{a}^\dagger_{\vec{k}'\lambda'}(0)]}$$

$\Rightarrow$ Equal-time commutators preserved by time evolution, but non-equal time are not.

Note: $$[\hat{a}^{\text{source}}_{\vec{k},\lambda}(t), \hat{a}^{\dagger\,\text{source}}_{\vec{k}',\lambda'}(t)] = g^*_{\vec{k}\lambda} g_{\vec{k}'\lambda'} \int_0^t dt' \int_0^t dt'' \overbrace{[\hat{\sigma}_-(t'), \hat{\sigma}_-(t'')]}^{\neq 0} e^{-i\omega_k(t'-t'')}$$

$$\neq \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}$$

(d) Plug solution for $\hat{a}_{\vec{k}\lambda}(t)$ into $\frac{d\hat{\sigma}_z}{dt} \Rightarrow$ Integro-differential equation

$$\Rightarrow \frac{d}{dt}\hat{\sigma}_z(t) = 2\Big[\sum_{\vec{k},\lambda}\left(i g_{\vec{k}\lambda}\hat{\sigma}_+(t)\hat{a}_{\vec{k}\lambda}(0)e^{-i\omega_k t} + h.c.\right) - \sum_{\vec{k},\lambda}\left(|g_{\vec{k}\lambda}|^2 \int_0^t dt' \hat{\sigma}_+(t)\hat{\sigma}_-(t')e^{-i\omega_k(t-t')} + h.c.\right)\Big]$$

Now take the expectation value in the state $|\Psi\rangle = |\psi_{atom}\rangle \otimes |Vac\rangle$ (constant in Heisenberg picture)

Note $\hat{a}_{\vec{k},\lambda}(0)|vac\rangle = 0$ $\qquad \langle vac|\hat{a}^\dagger_{\vec{k},\lambda} = 0$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{\sigma}_z(t)\rangle = -2\sum_{\vec{k},\lambda}|g_{\vec{k}\lambda}|^2\int_0^t dt'\left(\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle e^{-i\omega_k(t-t')} + c.c.\right)}$$

(e) Under Markov approx $\qquad \hat{\sigma}_-(t) = \hat{\Sigma}_-(t)\,e^{-i\omega_{eg}t}$

Slowly varying envelope

$$\Downarrow$$

$$\langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t')\rangle \approx \langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t)\rangle = \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle$$

$$\therefore \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle = \langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t')\rangle e^{i\omega_{eg}(t-t')}$$

$$\approx \langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t)\rangle e^{+i\omega_{eg}(t-t')} = \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle e^{+i\omega_{eg}(t-t')}$$

$$\therefore \frac{d}{dt}\langle\hat{\sigma}_z(t)\rangle \approx -2\sum_{\vec{k},\lambda}|g_{\vec{k}\lambda}|^2 \underbrace{\int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')}dt'}_{=f(\omega_{eg}-\omega_k)} + c.c.$$

$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle$

$$f(\omega_{eg}-\omega_k) \rightarrow \pi\delta(\omega_{eg}-\omega_k) - i\,\mathcal{P}\left[\frac{1}{\omega_{eg}-\omega_k}\right]$$

in usual Markov approx $t\to\infty$

$$\Rightarrow \frac{d}{dt}\langle\hat{\sigma}_z\rangle = -2 \underbrace{\sum_{\vec{k}\lambda} |g_{\vec{k}\lambda}|^2 \, 2\pi\,\delta(\omega_{\vec{k}} - \omega_{eg})}_{=\Gamma \text{ (spontaneous emission rate)}} \langle\hat{\sigma}_+\hat{\sigma}_-\rangle$$

Note: $\hat{\sigma}_+(t)\,\hat{\sigma}_-(t) = \dfrac{1+\hat{\sigma}_z(t)}{2}$ (projector)

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -\frac{\Gamma}{2}\left(1 + \langle\hat{\sigma}_z\rangle\right)}$$

Moral of the story: One can see spontaneous emission as arising from radiation reaction,

i.e. the oscillator decays because energy is carried away by the field. The vacuum part of $\hat{a}(t)$ played no role. It's role is to induce the initial dipole with a random phase, that then radiates.

Problem 3: Momentum and Angular Momentum in Field

From Maxwell's Equation

$$\vec{P} = \int d^3\vec{x} \; \frac{\vec{E}(\vec{x}) \times \vec{B}(\vec{x})}{4\pi c} = \vec{p}_{(\vec{x})} \quad \text{momentum density}$$

$$\vec{J} = \int d^3\vec{x} \; \left(\vec{x} \times \vec{p}_{(\vec{x})}\right)$$

Quantized field $\hat{\vec{A}}(\vec{x}) = \hat{\vec{A}}^{(+)}_{(\vec{x})} + \hat{\vec{A}}^{(-)}_{(\vec{x})}$

$$\hat{\vec{A}}^{(+)}(\vec{x}) = \sum_{\vec{k},\lambda}{}' \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \; \vec{\epsilon}_{\vec{k},\lambda} \, e^{i\vec{k}\cdot\vec{x}} \; \hat{a}_{\vec{k},\lambda}$$

$$\hat{\vec{A}}^{(-)} = \left(\hat{\vec{A}}^{(+)}\right)^{+}$$

$$\hat{\vec{E}}^{(+)} = i\sum_{\vec{k},\lambda}{}' \sqrt{\frac{2\pi\hbar\omega_k}{V}} \; \vec{\epsilon}_{\vec{k},\lambda} \, e^{i\vec{k}\cdot\vec{x}} \; \hat{a}_{\vec{k},\lambda}$$

$$\hat{\vec{B}}^{(+)} = i\sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar\omega_k}{V}} \left(\vec{e}_k \times \vec{\epsilon}_{\vec{k},\lambda}\right) e^{i\vec{k}\cdot\vec{x}} \; \hat{a}_{\vec{k},\lambda}$$

Notes: $\displaystyle\int_V d^3x \; \frac{e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}{V} = \delta_{\vec{k},\vec{k}'}$

$$\vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k},\lambda'} = \delta_{\lambda\lambda'}$$

(3a) Plug mode decomposition into $\hat{\vec{P}}$

$$\Rightarrow \hat{\vec{P}} = \int d^3x \left(\frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} + \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(+)}}{4\pi c} + h.c. \right)$$

Consider first term:

$$\int d^3x \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} = \sum_{\vec{k},\vec{k}',\lambda,\lambda'}' \frac{1}{4\pi c} \left(2\pi\hbar \sqrt{\omega_k \omega_{k'}}\right) \vec{\epsilon}_{\vec{k},\lambda} \times \left(\vec{e}_{\vec{k}'} \times \vec{\epsilon}^{\,*}_{\vec{k}',\lambda'}\right) \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{\delta_{\vec{k},\vec{k}'}} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k}',\lambda'}$$

$$= \sum_{\vec{k}} \sum_{\lambda,\lambda'}' \frac{\hbar\omega}{2c} \underbrace{\left[\vec{\epsilon}_{\vec{k},\lambda} \times \left(\vec{e}_{\vec{k}} \times \vec{\epsilon}^{\,*}_{\vec{k},\lambda'}\right)\right]}_{\vec{e}_{\vec{k}} (\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda'}) - \vec{\epsilon}^{\,*}_{\vec{k},\lambda'} (\vec{e}_{\vec{k}} \cdot \vec{\epsilon}_{\vec{k},\lambda})} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda'}$$

$$\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda'} = \delta_{\lambda\lambda'}, \qquad \vec{e}_{\vec{k}} \cdot \vec{\epsilon}_{\vec{k},\lambda} = 0$$

$$= \sum_{\vec{k}} \frac{\hbar \vec{k}}{2} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda}$$

having used $\vec{k} = \frac{\omega}{c} \vec{e}_{\vec{k}}$

by similar steps, using $\int \frac{d^3x}{V} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} = \delta_{\vec{k}', -\vec{k}}$

$$\int d^3x \, \frac{\vec{E}^{(+)} \times \vec{B}^{(+)}}{4\pi c} = \sum_{\vec{k}}' \sum_{\lambda\lambda'}' \frac{\hbar\omega}{2c} \underbrace{\left[\vec{\epsilon}_{\vec{k},\lambda} \times (\vec{e}_{-\vec{k}} \times \vec{\epsilon}_{-\vec{k},\lambda'})\right]}_{\vec{e}_{-\vec{k}} \, (\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{-\vec{k},\lambda'})} \hat{a}_{\vec{k},\lambda} \hat{a}_{-\vec{k},\lambda'}$$

<u>Aside</u>: $\vec{e}_{-\vec{k}} = -\vec{e}_{\vec{k}}$, thus by symmetry, when we sum over all $\vec{k}$,

$$\sum_{\lambda\lambda'} \vec{e}_{-\vec{k}} (\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{-\vec{k},\lambda'}) \xrightarrow[\vec{k} \to -\vec{k}]{} \sum_{\lambda\lambda'} \vec{e}_{\vec{k}} (\vec{\epsilon}_{-\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k},\lambda'})$$

$$= -\sum_{\lambda,\lambda'} \vec{e}_{-\vec{k}} (\vec{\epsilon}_{-\vec{k},\lambda'} \cdot \vec{\epsilon}_{\vec{k},\lambda})$$

Thus the terms cancel pairwise.

$$\Rightarrow \hat{\vec{P}} = \sum_{\vec{k},\lambda} \frac{\hbar\vec{k}}{2} (\hat{a}_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda} + h.c.)$$

$$= \sum_{\vec{k},\lambda} \hbar\vec{k} (\hat{a}^{\dagger}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} + \frac{1}{2})$$

But $\sum_{\vec{k}} \frac{\hbar\vec{k}}{2} = 0$ (vector's cancel)

$$\Rightarrow \boxed{\hat{\vec{P}} = \sum_{\vec{k},\lambda} \hbar\vec{k} \, (\hat{a}^{\dagger}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda})}$$

Neat! Momentum $= \hbar\vec{k} \times$ number of photons

(1b) Total angular momentum in field:

$$\hat{\vec{J}} = \int d^3x \;\; \vec{x} \times \hat{\vec{\mathcal{P}}}(\vec{x})$$

where $\hat{\vec{\mathcal{P}}}(\vec{x}) = \frac{1}{4\pi c} (\hat{\vec{E}} \times \hat{\vec{B}})$ = momentum density

Lets massage these equations a bit.

$$(\hat{\vec{E}} \times \hat{\vec{B}})_i = \epsilon_{ijk} E_j B_k \quad \text{(summation convention)}$$

$$= \epsilon_{ijk} E_j \epsilon_{klm} \partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_l A_m$$

$$= E_l \partial_i A_l - E_l \partial_l A_i$$

Now $\vec{J} = \int d^3x \;\; \vec{x} \times \vec{\mathcal{P}}(\vec{x})$

$$\Rightarrow J_j = \epsilon_{jki} \int d^3x \;\; x_k \, \mathcal{P}_i$$

$$= \epsilon_{jki} \frac{1}{4\pi c} \int d^3x \left[E_l (x_k \partial_i) A_l - (x_k E_l)(\partial_l A_i) \right]$$

$$= \frac{1}{4\pi c} \int d^3x \; E_l (\vec{x} \times \vec{\nabla})_j A_l$$

$$+ \frac{1}{4\pi c} \int d^3x \, \epsilon_{jki} \underbrace{\partial_l (x_k E_l)}_{\left[\delta_{lk} + \vec{\nabla} \cdot \vec{E}\right]} A_i \quad \left(\text{integration by parts}\right)$$

$\vec{\nabla} \cdot \vec{E} \to 0$ in free space

$$\Rightarrow J_j = \frac{1}{4\pi c}\left(\int d^3x \; E_\ell \, (\vec{x}\times\vec{\nabla})_j \, A_\ell \right.$$
$$\left. + \int d^3x \; (\vec{E}\times\vec{A})_j \right)$$

$$\Rightarrow \vec{J} = \vec{J}_{orb} + \vec{J}_{spin}$$

$$\boxed{\begin{aligned} \vec{J}_{orb} &= \frac{1}{4\pi c}\int d^3x \; E_\ell \, (\vec{x}\times\vec{\nabla}) \, A_\ell \\ \vec{J}_{spin} &= \frac{1}{4\pi c}\int d^3x \; (\vec{E}\times\vec{A}) \end{aligned}}$$

(1c)

Let us expand these terms in the plane wave basis:

$$\vec{J}_{orb} = \left(\frac{1}{4\pi c}\int d^3x \; E_\ell^{(-)} (\vec{x}\times\vec{\nabla}) A_\ell^{(+)} \quad + h.c. \right)$$
$$+ \left(\frac{1}{4\pi c}\int d^3x \; E_\ell^{(+)} (\vec{x}\times\vec{\nabla}) \vec{A}_\ell^{(+)} \quad + h.c. \right)$$

Consider

$$\frac{1}{4\pi c}\int d^3x \; E_\ell^{(-)} (\vec{x}\times\vec{\nabla}) A_\ell^{(+)}$$

$$= \sum_{\vec{k}\vec{k}',\lambda\lambda'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}} \; \hat{a}^\dagger_{\vec{k},\lambda} \hat{a}_{\vec{k}',\lambda'} \; \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k}',\lambda'} \qquad \begin{pmatrix} \text{Summing} \\ \text{over } \ell \end{pmatrix}$$

$$\underbrace{\int \frac{d^3x}{V} \, e^{-i\vec{k}\cdot\vec{x}} \, (\vec{x}\times -i\vec{\nabla}) \, e^{i\vec{k}'\cdot\vec{x}}}_{I}$$

Aside: $\int \frac{d^3x}{V}\, e^{-i\vec{k}\cdot\vec{x}} \left(\vec{x}\times -i\vec{\nabla}\right) e^{i\vec{k}'\cdot\vec{x}}$

$$= \int \frac{d^3x}{V}\, e^{-i\vec{k}\cdot\vec{x}} \left(\vec{x}\times \vec{k}'\right) e^{i\vec{k}'\cdot\vec{x}}$$

$$= \left[\int \frac{d^3x}{V}\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}\ \vec{x}\right] \times \vec{k}'$$

$$= -i\vec{\nabla}_{\vec{k}'} \underbrace{\left[\int \frac{d^3x}{V}\, e^{-i(\vec{k}-\vec{k}')}\right]}_{\delta_{\vec{k},\vec{k}'}} \times \vec{k}'$$

$$\Rightarrow \frac{1}{4\pi c}\int d^3x\ E_\ell^{(-)} \left(\vec{x}\times\vec{\nabla}\right) A_\ell^{(-)}$$

$$= \sum_{\vec{k},\vec{k}'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\ \hat{a}_i(\vec{k})\ \hat{a}_i^\dagger(\vec{k}') \left(-i\left(\vec{\nabla}_{k'}\, \delta(\vec{k}-\vec{k}')\right)\times\vec{k}'\right)$$

~~Integration~~ where $\hat{\vec{a}}(\vec{k}) \equiv \sum_\lambda \vec{\epsilon}_\lambda(\vec{k})\, \hat{a}_\lambda(\vec{k})$

Integration by parts

$$\Rightarrow \frac{1}{4\pi c}\int d^3x\ E_\ell^{(-)} \left(\vec{x}\times\vec{\nabla}\right) A_\ell^{(+)}$$

$$= \frac{\hbar}{2}\sum_k \hat{a}_i(\vec{k}) \underbrace{\left[+i\vec{k}\times\vec{\nabla}_{\vec{k}}\right]}_{\Uparrow} \hat{a}_i^\dagger$$

$\vec{k}$-space Orbital angular momentum operator

Consider

$$\int d^3x\ \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{-i\hbar}{2} \sum_{\vec{k}\vec{k}',\lambda\lambda'}' \hat{a}^{\dagger}_{\vec{k}\lambda}\, \hat{a}_{\vec{k}'\lambda'}\ \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \times \vec{\epsilon}_{\vec{k}',\lambda'}$$

$$\underbrace{\int \frac{d^3x}{V}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{\rightarrow\ \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$= \frac{-i\hbar}{2} \sum_{\vec{k}}' \left[\left(\vec{\epsilon}^{\,*}_{\vec{k},+} \times \vec{\epsilon}_{\vec{k},+}\right) \hat{a}^{\dagger}_{\vec{k}+} \hat{a}_{\vec{k}+} + \left(\vec{\epsilon}^{\,*}_{\vec{k},-} \times \vec{\epsilon}_{\vec{k},-}\right) \hat{a}^{\dagger}_{\vec{k}-} \hat{a}_{\vec{k}-} \right]$$

Aside $\vec{\epsilon}_{\vec{k},\pm} \equiv \dfrac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$ where $\vec{e}_1$ and $\vec{e}_2$ are two orthonormal vectors with $\vec{e}_1 \times \vec{e}_2 = \hat{k}$

$$\Rightarrow \vec{\epsilon}^{\,*}_{\vec{k},\pm} \times \vec{\epsilon}_{\vec{k},\pm} = \pm i\, \vec{e}_k$$

$$\Rightarrow \int d^3x\ \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}}' \left(\hat{a}^{\dagger}_{\vec{k},+} \hat{a}_{\vec{k},+} - \hat{a}^{\dagger}_{\vec{k},-} \hat{a}_{\vec{k},-} \right) \vec{e}_{\vec{k}}$$

Now $$\int d^3x\ \frac{\vec{E}^{(+)} \times \vec{A}^{(-)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}}' \left(\hat{a}_{\vec{k}+} \hat{a}^{\dagger}_{\vec{k},+} - \hat{a}_{\vec{k}-} \hat{a}^{\dagger}_{\vec{k}-} \right) \vec{e}_k$$

$$= \frac{\hbar}{2} \sum_{\vec{k}}' \left(\hat{a}^{\dagger}_{\vec{k}+} \hat{a}_{\vec{k}+} - \hat{a}^{\dagger}_{\vec{k}-} \hat{a}_{\vec{k}-} \right) \vec{e}_k$$ (commutator cancels)

Finally note: $\vec{\epsilon}_{\vec{k},\pm} \times \vec{\epsilon}_{\vec{k},\pm} = 0$

$$\Rightarrow \int d^3x\ \vec{E}^{(+)} \times \vec{A}^{(+)} = \int d^3x\ \vec{E}^{(-)} \times \vec{A}^{(-)}$$

$$= 0$$

Thus $$\boxed{\vec{J}_{spin} = \hbar \sum_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k},+}\hat{a}_{\vec{k},+} - \hat{a}^{\dagger}_{\vec{k},-}\hat{a}_{\vec{k},-}\right)\vec{e}_k}$$

Each photon has intrinsic "spin" angular momentum. In the circularly polarized, plane wave basis, the photon has a definite helicity, ~~carrying~~ carry one hbar of angular momentum along (opposite to) the direction of propagation $\vec{e}_k$ for positive (negative) handed polarization.

The photon is spin $S=1$, yet there are only two states with definite projection of angular momentum, whereas, we might expect three $(2S+1 = 3)$. This is a very subtle point coming from the fact the the photon is massless. For more details see,

"Photons and Atoms",

(1d) Mapping photon spin onto a two-state Hilbert space

Define $\vec{\hat{J}}_{spin} = \hat{J}_x \vec{e}_x + \hat{J}_y \vec{e}_y + \hat{J}_z \vec{e}_z$

where $\hat{J}_x = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_y = \frac{\hbar}{2i}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_z = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$

This is the Schwinger representation of angular momentum connecting the "Boson algebra" $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ to the angular momentum algebra $[\hat{J}_i, \hat{J}_j] = i\hbar\, \epsilon_{ijk} \hat{J}_k$

Check: $[\hat{J}_x, \hat{J}_y] = \frac{\hbar^2}{4i}\left([\hat{a}_+^\dagger \hat{a}_-, -\hat{a}_-^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_-]\right)$

$$= \frac{\hbar^2}{4i}\left[2\hat{a}_+^\dagger \hat{a}_+ \Big(\underbrace{[\hat{a}_-^\dagger, \hat{a}_-]}_{=-1}\Big) - 2\hat{a}_-^\dagger \hat{a}_- \Big(\underbrace{[\hat{a}_+^\dagger, \hat{a}_+]}_{=-1}\Big)\right]$$

$$= i\hbar\left(\frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)\right) = i\hbar \hat{J}_z \quad \checkmark$$

$$[\hat{J}_x, \hat{J}_z] = \frac{\hbar^2}{4}\Big([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-]\Big)$$

$$= \frac{\hbar^2}{4}\left(\hat{a}_+^\dagger \hat{a}_-(-1) - \hat{a}_+^\dagger \hat{a}_-(1) + \hat{a}_-^\dagger \hat{a}_+(1) - \hat{a}_-^\dagger \hat{a}_+(-1)\right)$$

$$= \frac{-\hbar^2}{2}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+) = -i\hbar \hat{J}_y \quad \checkmark$$

$$[\hat{J}_y, \hat{J}_z] = \frac{\hbar^2}{4i}\Big([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-]$$
$$- [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-]\Big)$$

$$= \frac{\hbar^2}{4i}\left(\hat{a}_+^\dagger \hat{a}_-(-1) - \hat{a}_+^\dagger \hat{a}_-(1) - \hat{a}_-^\dagger \hat{a}_+(1) + \hat{a}_-^\dagger \hat{a}_+(-1)\right)$$

$$= -\frac{\hbar^2}{2i}\left(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+\right)$$

$$= i\hbar\left[\frac{\hbar}{2}\left(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+\right)\right] = i\hbar \hat{J}_x \quad \checkmark$$

The Schwinger representation is the "second quantized form" of the spin ½ operators

$$\hat{J}_x = \frac{\hbar}{2}\left(|+_z\rangle\langle -_z| + |-_z\rangle\langle +_z|\right)$$
$$\hat{J}_y = \frac{\hbar}{2i}\left(|+_z\rangle\langle -_z| - |-_z\rangle\langle +_z|\right)$$
$$\hat{J}_z = \frac{\hbar}{2}\left(|+_z\rangle\langle +_z| - |-_z\rangle\langle -_z|\right)$$

"Second quantize" $|\pm_z\rangle \Rightarrow \hat{a}_\pm^\dagger$ create spin up or down

$\langle \pm_z| \Rightarrow \hat{a}_\pm$ annihilate spin up or down

thus, we can easily map the spin angular momentum of the ~~th~~ photon onto the Bloch sphere, also known as the Poincaré sphere as we visited in PS# 1